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A singular integral operator arising from 1/N expansions: analytical and numerical results

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Abstract. The spectrum of the singular integral operator

$$(K_\sigma\psi)(x)=\frac{1}{\pi}⨍_{-1}^{1}(x-y)^{-2}(\psi(x)-\psi(y))\sigma(\lambda,y)\,\mathrm{d}y$$

with $\sigma(\lambda,x)=(1-x^2)^{1/2}(1+\lambda^2x^2)^{1/2}$ is studied both analytically and numerically. It is shown that K_σ has a simple discrete spectrum $0=E_0(\lambda)<E_1(\lambda)<E_2(\lambda)<\ldots<E_n(\lambda)<\ldots$ with asymptotic behaviour

$$E_{n-1}(\lambda)\sim n\omega(\lambda)+q(\lambda)$$

where $\omega(\lambda)$ is interpreted as the frequency of the periodic orbits for a corresponding Hamiltonian dynamical system, and $q(\lambda)$ is an 'average potential' defined in terms of $\sigma(\lambda,x)$. Singular integral operators of this kind arise in the analysis of SU(N)-symmetric quantum dynamical systems and in two-dimensional SU(N) gauge theories in the limit $N\to\infty$.

1. Introduction

The object of this paper is to study the spectrum of the following singular integral operator

$$(K_\sigma\psi)(x)=\frac{1}{\pi}⨍_{-1}^{1}(x-y)^{-2}(\psi(x)-\psi(y))\sigma(\lambda,y)\,\mathrm{d}y \tag{1}$$

defined in a suitable domain in the real Hilbert space $L_2((-1,1),\sigma\,\mathrm{d}x)$, where

$$\sigma(\lambda,x)=(1-x^2)^{1/2}(1+\lambda^2x^2)^{1/2}\qquad 0\leqslant\lambda<1 \tag{2}$$

$$\|\psi\|^2=\frac{1}{\pi}\int_{-1}^{1}\psi(x)^2\sigma(\lambda,x)\,\mathrm{d}x \tag{3}$$

and $⨍$ denotes the Cauchy principal value. $\psi(x)$ will be assumed to be differentiable with $\psi'(x)$ satisfying a Holder condition in order that the Cauchy principal value in the definition (1) makes sense. In particular $\psi(x)$ must be bounded. Let $D_0(K_\sigma)$ denote such a domain in L_2.

The operator K_σ is met in the study of the planar ($N\to\infty$) limit of SU(N)-symmetric quantum dynamical systems (Marchesini and Onofri 1979). In particular the positive eigenvalues of $(1-\lambda^2)^{-1/2}K_\sigma$ represent the $N\to\infty$ limit of the energy levels of the

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SU(N)-symmetric Hamiltonian

$$H=-\tfrac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_{ij}\,\partial x_{ji}}+\tfrac{1}{2}\sum_{ij}x_{ij}x_{ji}+\frac{g}{N}\sum_{ijkh}x_{ij}x_{jk}x_{kh}x_{hi}\qquad (x_{ij}=x_{ji}^{*};\ i,j,k,h=1,\ldots,N)\tag{4}$$

belonging to the adjoint representation of SU(N) (after shifting to zero the value of the ground state). The coupling constant $g\in(0,+\infty)$ is a function of the parameter λ in equation (2) with $g(0)=0$ and $\lim_{\lambda\to 1} g(\lambda)=+\infty$.

A similar eigenvalue problem arises also in two-dimensional SU(N) gauge theory in the planar limit ('t Hooft 1974).

The results we present here are the following.

(i) K_σ is essentially self-adjoint; its unique self-adjoint extension has a purely discrete non-degenerate spectrum $0=E_0(\lambda)<E_1(\lambda)<\ldots<E_n(\lambda)<\ldots$ with $E_n(\lambda)\to+\infty$ when $n\to\infty$, and the corresponding eigenfunctions are of class C^∞.

(ii) $E_n(\lambda)$ grows asymptotically as $E_{n-1}(\lambda)=n\omega(\lambda)+q(\lambda)+\mathrm{O}(1/n)$, where $\omega(\lambda)$ and $q(\lambda)$ are defined in terms of $\sigma(\lambda,x)$; $\omega(\lambda)$ may be interpreted as the frequency of a related classical Hamiltonian system, while $q(\lambda)$ corresponds to the average potential.

(iii) The asymptotic value is actually reached very rapidly: the $\mathrm{O}(1/n)$ correction in the eigenvalue formula is of relative order 10^{-6} for n as low as $n=5$.

The result (i) is proved in § 2 by showing that $(K_\sigma+c)^{-1}$ is compact; point (ii) is also proved in § 2 where we show that the eigenvalue equation for K_σ can be reduced to the standard form of a singular integro-differential equation (Muskhelishvili 1953). Hence it can also be reduced to a regular Fredholm equation from which the asymptotic behaviour of eigenvalues and eigenfunctions can be easily obtained.

In § 3 we study K_σ from a numerical point of view to explore the low part of the spectrum which is not *a priori* covered by the asymptotic estimate. Several lines of attack are at our disposal. Surprisingly enough, the most simple one gives the best results. We apply a Gaussian quadrature formula to reduce K_σ to a finite $M\times M$ symmetric matrix $K^{(M)}$ with $M=50,60,70,80,90$. The eigenvalues $E_n^{(M)}(\lambda)$ are then *extrapolated* in the variable $1/M$ to the formal value $1/M=0$. The asymptotic behaviour which is known analytically is reproduced to a very high accuracy, while it is completely masked in the finite approximations $E_n^{(M)}$.

2. The spectrum of K_σ and its asymptotic behaviour

The first property of K_σ to be proved is the following.

Theorem 1. K_σ is symmetric and positive semi-definite.

Proof. For any ψ and ϕ belonging to $D_0(K_\sigma)$ it holds that

$$\int \sigma(\lambda,x)\phi(x)\,⨍\,(x-y)^{-2}(\psi(x)-\psi(y))\sigma(\lambda,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$=\lim_{\epsilon\to 0}\int\int_{|x-y|>\epsilon}\sigma(\lambda,x)\sigma(\lambda,y)\phi(x)(\psi(x)-\psi(y))(x-y)^{-2}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\tfrac{1}{2}\int\int\sigma(\lambda,x)\sigma(\lambda,y)(\phi(x)-\phi(y))(\psi(x)-\psi(y))(x-y)^{-2}\,\mathrm{d}x\,\mathrm{d}y$$

which is symmetric under interchange of ψ and ϕ.

We also have

$$(\psi, K_\sigma\psi) = \tfrac{1}{2}\iint \sigma(\lambda, x)\sigma(\lambda, y)\left(\frac{\psi(x)-\psi(y)}{x-y}\right)^2 \frac{dx}{\pi}\frac{dy}{\pi} \geqslant 0 \tag{5}$$

with $(\psi, K_\sigma\psi) = 0$ implying $\psi(x) = \text{constant}$, which is the first (trivial) eigenfunction.

The next property we want to prove is the following.

Theorem 2. K_σ has a unique self-adjoint extension with a purely discrete non-degenerate spectrum extending to $+\infty$.

To prove theorem 2 we need some lemmas.

Lemma 1. K_σ is equivalent under a similarity transformation to the operator H_σ given by

$$(H_\sigma\phi)(x) = q(\lambda, x)\phi(x) - \frac{1}{\pi}\sigma(\lambda, x) ⨍_{-1}^{1} \frac{\phi'(y)}{y-x}\,dy \tag{6}$$

where $\phi \in L_2((-1, 1), \sigma^{-1}\,dx)$ is of the form $\phi = \sigma(\lambda, x)\psi(x)$, $\psi \in D_0(K_\sigma)$. The 'potential' $q(\lambda, x)$ is defined by

$$q(\lambda, x) = \frac{1}{\pi}\frac{d}{dx} ⨍_{-1}^{1} (y-x)^{-1}\sigma(\lambda, y)\,dy. \tag{7}$$

Proof. Starting from the definition (1), we add and subtract the quantity $\sigma(\lambda, x)\psi(x)(x-y)^{-2}$ under the integral sign to obtain

$$\begin{aligned}(K_\sigma\psi)(x) = \psi(x) &⨍_{-1}^{1} (x-y)^{-2}(\sigma(\lambda, y) - \sigma(\lambda, x))\,dy/\pi \\ &- \int_{-1}^{1} (x-y)^{-2}(\sigma(\lambda, y)\psi(y) - \sigma(\lambda, x)\psi(x))\,dy/\pi.\end{aligned}$$

We now apply the following identity, which can easily be established through an integration by parts:

$$⨍_{-1}^{1} (x-y)^{-2}(f(y)-f(x))\,dy = \frac{d}{dx}\int_{-1}^{1} (y-x)^{-1}f(y)\,dy + 2f(x)/(1-x^2).$$

The result then follows, by defining $\phi(x) = \sigma(\lambda, x)\psi(x)$.

Lemma 2. $q(\lambda, x)$ is bounded in $|x| \leqslant 1$.

Proof. The Taylor series of $(1+\lambda^2y^2)^{1/2}$ is uniformly convergent in $|y| \leqslant 1$; we then integrate term by term by applying the well known result

$$⨍_{-1}^{1} (y-x)^{-1}(1-y^2)^{1/2}y^{2k}\,dy = -\sum_{0}^{k}\binom{\frac{1}{2}}{h}(-1)^h x^{2(k-h)+1}.$$

The power series that we obtain is convergent for $|x| < \lambda^{-1}$ and it can be resummed to yield

$$q(\lambda, x) = -{}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; -\lambda^2) - \frac{\lambda^2}{\pi}\frac{d}{dx}\int_{-1}^{1}\frac{x^3\sigma(\lambda, y)}{1+\lambda^2x^2y^2}\,dy$$

and the result follows (notice that even the limit $\lambda \to 1$ gives a bounded potential).

Lemma 3. In the special case $\lambda = 0$, $H_{\sigma(0)}$ has the spectral decomposition

$$(H_{\sigma(0)}\phi)(x) = \sum_{n=0}^{\infty} nU_n(x)(1-x^2)^{1/2} \int_{-1}^{1} U_n(y)\phi(y)\frac{\mathrm{d}y}{\pi} \tag{8}$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind (suitably normalised).

Proof. This is a simple consequence of the well known properties of Chebyshev polynomials (Abramowitz and Stegun 1965) together with $q(0, x) \equiv -1$ (from lemma 2). This fact was independently proved by Calogero and Perelomov (1978).

Now let us proceed to the proof of theorem 2. Let us decompose H_σ into the sum $q(\lambda, x) + H_\sigma^{(0)}$ where

$$(H_\sigma^{(0)}\phi)(x) = -\sigma(\lambda, x) \int\!\!\!\!\!\!- _{-1}^{1} (y-x)^{-1}\phi'(y)\,\mathrm{d}y/\pi.$$

Let us introduce a orthonormal basis in $L_2((-1, 1), \sigma^{-1}\,\mathrm{d}x)$ given by

$$\phi_n(x) = \chi^{1/2}(x)(1-x^2)^{1/2}U_n(x) \tag{9}$$

where $\chi(x) = (1+\lambda^2x^2)^{1/2}$. According to lemma 3, the functions ϕ_n satisfy the eigenvalue equation

$$(\chi^{-1/2}H_\sigma^{(0)}\chi^{-1/2}\phi_n)(x) = (n+1)\phi_n(x).$$

It follows that

$$(H_\sigma^{(0)}\phi)(x) = \chi^{1/2}(x) \sum_{n=0}^{\infty} (n+1)\phi_n(x)(\phi_n, \chi^{1/2}\phi). \tag{10}$$

$H_\sigma^{(0)}$ then admits a unique self-adjoint extension, with domain D prescribed by the spectral decomposition (10)†. Moreover $H_\sigma^{(0)}$ has a *compact inverse*

$$(H_\sigma^{(0)})^{-1} = \chi^{-1/2} \sum_{n=0}^{\infty} (n+1)^{-1}\phi_n(\phi_n, .)\chi^{-1/2}$$

(recall that both $\chi(x)$ and $\chi(x)^{-1}$ are bounded). It follows that $H_\sigma^{(0)}$ has a purely discrete spectrum extending to $+\infty$. This same property holds for $H_\sigma = H_\sigma^{(0)} + q(\lambda, x)$ since $q(\lambda, x)$ is bounded. In fact we have for any $c > 0$

$$(H_\sigma^{(0)} + q + c)^{-1} = (1 + H_\sigma^{(0)\,-1}(q+c))^{-1}H_\sigma^{(0)-1}$$

which shows that $H_\sigma^{(0)} + q$ has a compact resolvent. That the spectrum is non-degenerate is quite obvious for sufficiently small λ, but this property will be proved in general after theorem 3.

It should be noticed that the domain of self-adjointness of H_σ is given by

$$D(H_\sigma) = \left\{\psi(x) = \sum_{n=0}^{\infty} \alpha_n U_n(x)(1-x^2)^{1/2}, \sum_{n=0}^{\infty} (n+1)^2\alpha_n^2 < \infty\right\}$$

and correspondingly

$$D(K_\sigma) = \left\{\psi(x) = \sum_{n=0}^{\infty} \alpha_n U_n(x), \sum_{n=0}^{\infty} (n+1)^2\alpha_n^2 < \infty\right\}$$

† In other words, the closure $\bar{H}_\sigma^{(0)}$ is self-adjoint in D and $\sigma D_0(K_\sigma)$ is a core for $\bar{H}_\sigma^{(0)}$.

which are proper extensions of the original domains we started with. Nevertheless the eigenfunctions of K_σ can be easily shown to be C^∞ in $(-1, 1)$ and the action of K_σ on its eigenfunctions is correctly represented by the singular integral operator (1). (Hint: the components α_n of the eigenfunctions decrease faster than any power n^{-k} for $n \to \infty$.)

Let $E_n(\lambda)$ be the nth eigenvalue of K_σ (and of H_σ). We know from theorem 2 that $E_n(\lambda) \to \infty$ as $n \to \infty$. We can now characterise the asymptotic behaviour of $E_n(\lambda)$.

Theorem 3. The asymptotic expansion of $E_n(\lambda)$ is given by

$$E_n(\lambda) = (n+1)\omega(\lambda) + q(\lambda) + \mathrm{O}(1/n) \tag{11}$$

where

$$\omega(\lambda) = \left(\frac{1}{\pi}\int_{-1}^{1} \sigma(\lambda, x)^{-1}\, \mathrm{d}x\right)^{-1} = \frac{\pi(1+\lambda^2)^{1/2}}{2K[\lambda^2(1+\lambda^2)^{-1}]} \tag{12}$$

($K(.)$ being the complete elliptic integral of the first kind) and

$$q(\lambda) = \frac{1}{\pi}\omega(\lambda)\int_{-1}^{1} \sigma(\lambda, x)^{-1} q(\lambda, x)\, \mathrm{d}x. \tag{13}$$

Proof. The eigenvalue equation for E_n can be cast into the form

$$\frac{\phi_n(x)}{B(x)} - \frac{1}{\pi}\int_{-1}^{1} \frac{\phi_n'(y)}{y-x}\,\mathrm{d}y = 0 \qquad \phi_n(1) = \phi_n(-1) = 0 \tag{14}$$

where $B(x) = \sigma(\lambda, x)/(q(\lambda, x) - E_n)$. Equation (14) is studied in detail in Muskhelishvili's book (1953) to which we refer for details. Since $B(x)$ is even in x, the eigenfunctions can always be chosen of definite parity. Let us first consider the case $\phi_n(-x) = \phi_n(x)$. Equation (14) is then equivalent to the regular Fredholm equation

$$\phi_n(x) + \int_{-1}^{1} H(x, y)\phi_n(y)\,\mathrm{d}y = \phi_n(0)\cos\tau(x) \tag{15}$$

where

$$\tau(x) = \int_0^x B(y)^{-1}\,\mathrm{d}y \tag{16}$$

$$H(x, y) = \int_0^x (1-z^2)^{-1/2} R(z, y)\cos(\tau(x) - \tau(z))\,\mathrm{d}z \tag{17}$$

$$R(x, y) = (y-x)^{-1}[\chi(y)^{-1}(q(\lambda, y) - E_n) - \chi(x)^{-1}(q(\lambda, x) - E_n)]. \tag{18}$$

We know from theorem 2 that equation (15) has a solution when E_n belongs to the spectrum; from the theory of Fredholm equations this solution is unique and the spectrum is non-degenerate. Since $q(\lambda, x)$ is bounded, $q(\lambda, x) - E_n$ does not vanish in $[-1, 1]$ for sufficiently large n; this fact allows us to integrate by parts in equation (17) to obtain

$$H(x, y) = \frac{R(0, y)}{q(\lambda, 0) - E_n}\sin\tau(x) + \mathrm{O}(E_n^{-1}). \tag{19}$$

The leading term of $H(x, y)$ as $n \to \infty$ is *factorised* and *odd in* y so that we obtain the explicit asymptotic behaviour of the eigenfunctions

$$\phi_n(x) \sim \phi_n(0)\cos\tau(x). \tag{20}$$

The boundary condition $\phi_n(1)=0$ gives then equation (11) with odd n. A similar argument holds for even n.

The asymptotic estimate of theorem 3 turns out to be very accurate (10^{-8} relative error) for $n \geqslant 5$, but it gives a good approximate value for $E_n(\lambda)$ also for n as low as $n=2$ (see table 1 for a comparison with the numerical results of § 3)†.

It is obvious that the case $\lambda=0$ is rather exceptional, since $q(0,x)\equiv -1$ and $H(x,y)\equiv 0$. It follows that equation (20) gives the exact solution in terms of Chebyshev polynomials of the second kind with $E_n(0)=n$, as anticipated in lemma 3.

Table 1. A: $(1-\lambda^2)^{-1/2}E_n(\lambda)$, numerical value (§ 3). B: $(1-\lambda^2)^{-1/2}E_n(\lambda)$, from asymptotic estimate (11). C: $(E_{n+1}-E_n)\omega(\lambda)^{-1}$.

	n	A	B	C	
$\lambda^2=0{\cdot}1$	1	1·05424757	1·05470189	0·97646	
	2	2·13437110	2·13436573	1·00043	$\omega(\lambda)=1{\cdot}02425905$
	3	3·21402951	3·21402957	0·99999	$q(\lambda)=-1{\cdot}04794004$
	4	4·29369340	4·29369341	$1+5\times10^{-8}$	
	5	5·37335724	5·37335725	$1+10^{-9}$	
$\lambda^2=0{\cdot}2$	1	1·11865524	1·12043877	0·95548	
	2	2·29126038	2·29122024	1·00156	$\omega(\lambda)=1{\cdot}04717878$
	3	3·46200102	3·46200171	0.99997	$q(\lambda)=-1{\cdot}09220666$
	4	4·63278317	4·63278318	$1+6\times10^{-7}$	
	5	5·80356462	5·80356465	$1-2\times10^{-8}$	
$\lambda^2=0{\cdot}4$	1	1·29357774	1·30074098	0·91954	
	2	2·70780820	2·70751414	1·00530	$\omega(\lambda)=1{\cdot}08968180$
	3	4·11427816	4·11428729	0·99978	$q(\lambda)=-1{\cdot}17181395$
	4	5·52106066	5·52106044	1·00001	
	5	6·92783352	6·92783359	$1-2\times10^{-7}$	
$\lambda^2=0{\cdot}8$	1	2·25098376	2·290387	0·86443	
	2	4·89702771	4·894395	1·01614	$\omega(\lambda)=1{\cdot}16454786$
	3	7·49813389	7·498404	0·99889	$q(\lambda)=-1{\cdot}30480341$
	4	10·10229667	10·102412	1·00006	
	5	12·70629728	12·706420	$1-3\times10^{-6}$	

3. Numerical calculation of the eigenvalues

We shall now apply a discrete approximation to the eigenvalue equation for K_σ. Since $\sigma(\lambda,x)$ is singular at the endpoints, a discrete approximation is best performed by introducing the variable $\theta=\cos^{-1}x$. The Riemann sum in θ is equivalent to a Gauss quadrature formula in terms of Chebyshev polynomials of the second kind. We find that K_σ is approximated by the $M\times M$ real symmetric matrix $K^{(M)}$ with entries

$$\begin{aligned} K_{ij}^{(M)} &= -(M+1)^{-1}\frac{(1-x_i^2)^{1/2}(1-x_j^2)^{1/2}\chi(x_i)^{1/2}\chi(x_j)^{1/2}}{(x_i-x_j)^2} \qquad (i\neq j) \\ K_{ii}^{(M)} &= (M+1)^{-1}\sum_{k\neq i}{}_1^M\frac{(1-x_k^2)\chi(x_k)}{(x_k-x_i)^2} \end{aligned} \tag{21}$$

where $x_j=\cos[k\pi/(M+1)]$ are the zeros of $U_M(x)$ and $\chi(x)$ is the same as in § 2.

† It turns out that $E_1(\lambda)$ is closely approximated by the variational bound $E_1\leqslant(\psi,K_\sigma\psi)/(\psi,\psi)$ with $\psi(x)\equiv x$.

The eigenvalues $E_n^{(M)}$ of $K^{(m)}$ have been calculated for $M=50, 60, 70, 80, 90$ and for several values of $\lambda \in [0, 1)$. Truncation errors become severe for $n \gtrsim \frac{1}{2}M$, culminating in complex eigenvalues (this is quite a general phenomenon, see Ralston (1965)).

Let us recall that the eigenvalues of K_σ for $\lambda=0$ are simply given by $E_n(0)=n$ $(n=0, 1, 2, \ldots)$. A remarkable feature of the numerical analysis at $\lambda=0$ is that all the eigenvalues $E_n^{(M)}(0)$ are interpolated by the simple formula†

$$E_n^{(M)}(0)=\frac{Mn-\frac{1}{2}n^2}{M+1} \qquad (n=0, 1, \ldots, M-1)$$

which gives in fact the exact solution in the limit $M\to\infty$. This fact suggests that we look for an interpolation formula also for $\lambda>0$. A Langrangian interpolation formula in the variable $1/M$ (at fixed n) gives the results reported in table 1, where the differences $E_{n+1}(\lambda)-E_n(\lambda)$ are seen to approach very quickly the asymptotic value $\omega(\lambda)$. This property is, however, completely masked at every finite approximation, where the second differences $E_{n+1}-2E_n+E_{n-1}$ are approximately constant and different from zero. We made a check of the interpolation formula by calculating $E_n^{(90)}(\lambda)$ starting from previous values from $M=50$ to $M=80$. The agreement is very good for $n\leqslant 10$ (see table 2). All this evidence supports the belief that our approximation scheme gives the correct values of the first ten eigenvalues. For $n\geqslant 5$ the asymptotic estimate of theorem 3 can be safely applied with 5–8 figure accuracy (depending on λ).

Table 2. Sample results on the accuracy of Lagrange interpolation. A: $(E_{n+1}^{(90)}-E_n^{(90)})(1-\lambda^2)^{-1/2}$. B: the same quantity obtained by Lagrange interpolation from the values at $M=50$ through 80. C: relative error of B with respect to A.

	n	A	B	C
$\lambda^2=0{\cdot}2$	1	1·10030572	1·10030570	2×10^{-8}
	5	1·09919804	1·09919804	4×10^{-9}
	10	1·03407320	1·03407322	2×10^{-8}
	15	0·96892981	0·96893002	2×10^{-7}
	20	0.91679703	0·91679808	1×10^{-6}
$\lambda^2=0{\cdot}8$	1	2·21555160	2·21555162	1×10^{-8}
	5	2·44402962	2·44402964	1×10^{-8}
	10	2·29747397	2·29747438	4×10^{-7}
	15	2·15047209	2·15047755	3×10^{-6}
	20	2·03243814	2·03246788	1×10^{-5}

Of course, other numerical methods can be devised to solve the eigenvalue problem. For example, one can expand $\psi(x)$ into Chebyshev polynomials of the second kind and truncate the series. One could also study the equivalent Fredholm equation (15). 't Hooft (1974) and Einhorn (1976) report similar calculations performed on a related eigenvalue problem. The method adopted here is rather fast: an overall 18 s of computer time to calculate the first twenty eigenvalues for eight different values of λ.

† These are actually the exact eigenvalues of $K^{(M)}$ $(\lambda=0)$; this result is a special case of a theorem on the zeros of Jacobi polynomials (Ahmed *et al* 1979).

4. Concluding remarks

Let us conclude with a few remarks.

(a) It is clear that all qualitative results (theorems 1, 2 and 3) proven in the case $\chi(x)=(1+\lambda^2x^2)^{1/2}$ $(0\leq\lambda<1)$ apply also in general for any $\chi(x)$, provided that $\chi(x)$ and $\chi(x)^{-1}$ are smooth, bounded and such that the corresponding potential $q(x)$ of lemma 1 be bounded.

(b) One can consider the case of a singular integral operator of the type of equation (6) with unbounded $q(x)$. Such a problem arises in the planar limit of a two-dimensional Yang–Mills theory ('t Hooft 1974). Provided that $q(x)$ is bounded from below, theorem 2 still applies but the asymptotic behaviour of the eigenvalue has a logarithmic correction, $E_n\sim n\omega+\beta\ln n$, as can be shown by the WKB approximation ('t Hooft 1974).

(c) It would be desirable to relate our results to the general theory of pseudo-differential operators (PDO). For instance, properties similar to those proven in theorems 2 and 3 are shared by a very general class of PDO acting on compact boundaryless manifolds (Duistermaat and Guillemin 1975). In fact the asymptotic spectrum of K_σ can be intuitively understood in terms of the WKB approximation. The symbol of K_σ as a PDO is given to leading order by $K(x,p)=\sigma(\lambda,x)|p|+q(\lambda,x)$. This is to be interpreted as a classical Hamiltonian with reflecting barriers at $x=\pm1$. The semiclassical estimate $\oint p\,dx=2\pi n$ reproduces the asymptotic result of theorem 3. Notice that $\omega(\lambda)$ is the frequency of the classical orbits and $q(\lambda)$ is just the average of the potential $q(\lambda,x)$ along any orbit.

(d) We recall that the spectrum of the operator $(1-\lambda^2)^{-1/2}K_\sigma$ coincides with the $N\to\infty$ limit of the spectrum of the anharmonic SU(N)-symmetric oscillator (4) restricted to the adjoint representation. From this point of view the results embodied in theorems 1, 2 and 3 show the consistency of the planar limit. The limit $\lambda\to0$ corresponds to the harmonic oscillator with frequency $\omega(0)=1$, which 'explains' the result of lemma 3, $E_n(0)=n$. The fact that also for $\lambda>0$ the spectrum is very nearly reproduced by $E_{n-1}(\lambda)\sim n\omega(\lambda)+q(\lambda)$ shows that the anharmonic interaction in the planar limit simply amounts to a renormalisation of the frequency.

(e) As far as the convergence of the numerical method of § 3 is concerned, we notice that it has been proved that the algorithm converges to the exact solution for $\lambda=0$. It would be nice to have a proof of convergence also for $\lambda>0$. In a related inhomogeneous problem, such a proof can be found in the literature (Kalandiya 1975).

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